Embedded robotics, class

List 1

1. Show that the maps defined below are local diffeomorphisms in a neighbourhood of 0:

   a). $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x) = (x_3, x_2, x_1 - \sin x_2)$

   b). $\varphi: \mathbb{R}^4 \to \mathbb{R}^4$, $\varphi(x) = (x_1 x_2, -x_3 \sin x_1 + x_4 \cos x_1 - x_2, x_3 \cos x_1 + x_4 \sin x_1)$

   c). $\varphi: \mathbb{R}^5 \to \mathbb{R}^5$, $\varphi(x) = (x_1 \sin x_2, \cos x_2 \sin x_3, x_4, x_5 + x_4^3 - x_1^{10})$

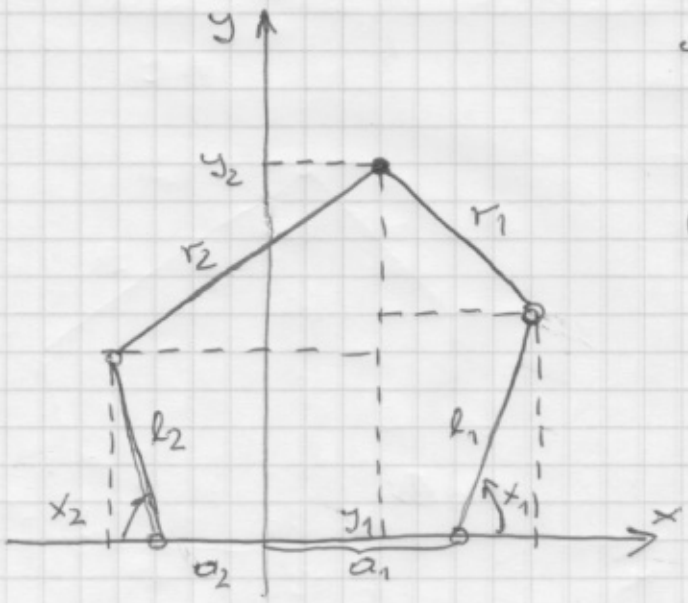
   Are these diffemorphisms global?

2. Prove that the system of equations $x_1 y_1 - x_2 y_2 = 0$, $x_2 y_1 + x_1 y_2 = 2$ defines a function $y = g(x)$. Compute the derivative $Dg(x)$ at $x_1 = x_2 = y_1 = y_2 = 1$.

3. For the kinematics of a double pendulum, given as $y_1 = l_1 \cos x_1 + l_2 \cos(x_1 + x_2)$, $y_2 = l_1 \sin x_1 + l_2 \sin(x_1 + x_2)$ show the existence of inverse kinematics outside singular configurations.

4. Using the Implicit Function Theorem find conditions under which the eigenvalues of a matrix $\underset{n \times n}{A}$ are a function of coefficients of its characteristic equation.

5. Check the existence of the forward and inverse kinematics of the mechanism displayed below and described as follows:



$$(a_1 + l_1 \cos x_1 - y_1)^2 + (y_2 - l_1 \sin x_1)^2 = r_1^2$$

$$(a_2 + y_1 + l_2 \cos x_2)^2 + (y_2 - l_2 \sin x_2)^2 = r_2^2.$$

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List 2

1. For a rotation matrix $R \in SO(3)$ compute the norms $\|R\|_2$ and $\|R\|_F$. Find also the norm $\|R\|_1$ for $R = Rot(z_\alpha)$.
2. For a matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ compute the norms $\|A\|_1$, $\|A\|_2$, $\|A\|_F$ and $\|A\|_\infty$ as well as the condition number $\varkappa(A)$.
3. Demonstrate that every operator matrix norm has the ring property $\|AB\| \leq \|A\| \|B\|$,
4. Show that $\|A\|_F^2 = tr(AA^T)$, $A_{n\times n}$.
5. Show that for a square matrix $A_{n\times n}$ there holds

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_1 \leq \sqrt{n} \|A\|_F .$$

Hint: Use the inequality $\left(\sum_{i=1}^{n} |a_i|\right)^2 \leq n \sum_{i=1}^{n} |a_i|^2$.

6. Given a system of linear equations $Ax = b$, $A_{n\times n}$, with perturbed right hand side, so that $A\hat{x} = b + \varepsilon$. Show that the relative error $\delta x = \frac{\|\hat{x} - x\|}{\|x\|}$ of the solution satisfies the condition

$$\delta b \leq \delta x \leq \varkappa(A) \delta b ,$$

where $\delta b = \frac{\|\varepsilon\|}{\|b\|}$ and $\varkappa(A)$ denotes the condition number of A.

7. Using the theorems on submersions and immersions find the normal forms of the following functions:

a). $f(x) = x_1 + x_2^2$, $x = (x_1, x_2) \in \mathbb{R}^2$,

b). $f(x) = (\sin x, \cos x)$, $x \in \mathbb{R}$,

c). $f(x) = (x, tg\, x)$, $x \in \mathbb{R}$,

d). $f(x) = (x_1 + x_2^2, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, $f(x) \in \mathbb{R}^2$.

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List 3

1. Check the existence and (non)degeneracy of critical points of the following functions:

a). $f(x) = x^3$, $\quad x \in \mathbb{R}$

b). $f(x) = x_1^3 - 3x_1^2 x_2$, $\quad x = (x_1, x_2) \in \mathbb{R}^2$

c). $f(x) = x_1^2$, $\quad x = (x_1, x_2) \in \mathbb{R}^2$

d). $f(x) = x_1 x_2$, $\quad x = (x_1, x_2) \in \mathbb{R}^2$

e). $f(x) = x_1^2 \cos x_2 + \sin^2 x_2$, $\quad x = (x_1, x_2) \in \mathbb{R}^2$

2. Show that the function

$$f(x) = x_1^2 + x_1 x_2 + x_2^2$$

is LR-equivalent to $g(x) = x_1^2 + x_2^2$.

3. Show that the function

$$f(x) = x_1 x_2 + x_2^2$$

is LR-equivalent to $g(x) = x_1^2 - x_2^2$.

Suggestion: In 2 and 3 do not use the Morse theorem.

4. Using the Morse theorem discover the normal forms of the following functions around (00):

a). $f(x) = x_1^2 \cos x_2 + \sin^2 x_2$

b). $f(x) = \cos x_1 - 2x_1 x_2 + \cos x_2$

c). $f(x) = x_1 \sin x_2 + x_2 \sin x_1$

d). $f(x) = x_1^2 \cos x_3 + x_2 x_3 + x_3^2$

e). $f(x) = \sin x_1 \sin x_2 - x_3^2$

f). $f(x) = x_1 x_2 + x_2 x_3 - x_1 x_3$

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List 4

1. Using the Fixed point theorem derive a sufficient condition for convergence of the following algorithm solving a system of linear equations $x = Ax$, $x \in \mathbb{R}^n$.
The algorithm: $x_{k+1} = Ax_k$, x_0 – initial point.

2. Show that the fundamental matrix $\phi(t,s)$ of the system $\dot{x} = A(t)x$ satisfies the identity

$$\frac{\partial \phi^T(s,t)}{\partial t} = -A^T(t)\phi^T(s,t).$$

3. Check that for a constant matrix $A(t) = A$ the Peano-Baker series gives $\phi(t,0) = e^{tA}$.

4. Check that a matrix $M(t) = \int_0^t \phi(t,s)B(s)B^T(s)\phi^T(t,s)\,ds$ satisfies a Lyapunov equation

$$\dot{M} = B(t)B^T(t) + A(t)M(t) + M(t)A^T(t).$$

5. Prove the Gronwall-Bellman lemma. Hint: use the fact that $\frac{\phi(t)}{a\int_0^t \phi(s)\psi(s)\,ds + b} \leq 1$.

6. Examin the asymptotic stability of the point 0 in the following linear systems invoking the Wazewski inequality:

a). $\dot{x} = -tx$, $\dot{y} = -y$

b). $\dot{x} = -x + \frac{2y}{1+t^2}$, $\dot{y} = -y$

c). $\dot{x} = -2x + 2y\sin t$, $\dot{y} = -2y$

d). $\dot{x} = -t^2x + y\cos t$, $\dot{y} = -t^2y - x\cos t$.

Embedded, class

List 5

1. Check stability of the following systems

a). $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}-1 & e^{\frac{1}{2}t}\\ 0 & -1\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$, b). $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}-10 & e^{3t}\\ 0 & -2\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$,

c). $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}-1 & e^{2t}\\ 0 & -2\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$, d). $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}-1 & 2\sin t\\ 0 & -(t+1)\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$.

2. Show that the point $(0,0)$ in the system

$\dot{x}_1=-x_1+x_2+(x_1^2+x_2^2)\sin t$, $\dot{x}_2=-x_1-x_2+(x_1^2+x_2^2)\cos t$

is exponentially stable and estimate its stability region.

Hint: Take $V(x)=x_1^2+x_2^2$.

3. Examine stability of the point $(0,0)$ of the system

$\dot{x}_1=h(t)x_2-g(t)x_1^3$, $\dot{x}_2=-h(t)x_1-g(t)x_2^3$, $g(t), h(t)$ smooth and bounded, $g(t)\geq k>0$.

Hint: Use $V(x)=x_1^2+x_2^2$.

4. Let $V(x)$ denote a smooth potential function. Demonstrate that the gradient system

$$\dot{x}=-\frac{\partial V(x)}{\partial x}=-DV(x)$$

has no closed orbits.

5. Let $H(x,y)$, $x,y\in\mathbb{R}^n$, denote a smooth Hamiltonian. Show that the Hamiltonian system

$$\dot{x}=\frac{\partial H(x,y)}{\partial y},\quad \dot{y}=-\frac{\partial H(x,y)}{\partial x}$$

cannot have an asymptotically stable equilibrium point.

Embedded, class

List 6

1. Show that the dynamic system

$$\dot{x} = -\lambda y + xy, \quad \dot{y} = \lambda x + \tfrac{1}{2}(x^2 - y^2), \quad x, y \in \mathbb{R}, \ \lambda > 0 \text{ a parameter,}$$

is Hamiltonian. Define the Hamiltonian and draw a phase portrait of this system.

2. Prove that the dynamic system

$$\dot{x} = x^2 - y^2, \quad \dot{y} = 2x(x^2 - y), \quad x, y \in \mathbb{R}$$

has a first integral.

3. Find a first integral of the dynamic system

$$\dot{x} = y, \quad \dot{y} = x - 2x^3$$

and draw its phase portrait.

4. Find a first integral of the Lotka-Volterra system

$$\dot{x} = ax - bxy, \quad \dot{y} = -cy + bxy, \quad x, y \geq 0, \quad a, b, c > 0$$

and draw a phase portrait of this system.

5. Check stability of the equilibrium point $(x_0, y_0) = (0, 0)$ of the system

$$\dot{x} = -y + x(x^2+y^2)\sin\sqrt{x^2+y^2}, \quad \dot{y} = x + y(x^2+y^2)\sin\sqrt{x^2+y^2}.$$

Hint: Use the polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$.

6. Show that the Lie bracket

$$[X, Y](x) = \frac{d}{dt}\Big|_{t=0} Ad_x^t Y(x) = DY(x)X(x) - DX(x)Y(x).$$

Embedded, class

List 7

1. Using the lecture notes show that the composition of flows of commuting vector fields is commutative:
$[X,Y]=0 \Rightarrow \varphi_t(\psi_s(x)) = \psi_s(\varphi_t(x))$.

2. Show that for any $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$

$$L_{[X,Y]} f = L_X(L_Y f) - L_Y(L_X f).$$

3. Using the property $[fX, gY] = fg[X,Y] + f L_X g Y - g L_Y f X$ prove that a distribution $\mathcal{D} = \operatorname{span}_{C^\infty(\mathbb{R}^n,\mathbb{R})}\{X_1(x), \dots, X_k(x)\}$ is involutive if and only if, for every i,j; $[X_i, X_j] \in \mathcal{D}$.

4. Check involutivity of the following distributions:

a). $\mathcal{D}_1 = \operatorname{span}_{C^\infty(\mathbb{R}^3,\mathbb{R})}\left\{\begin{pmatrix}1\\x_2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$,

b). $\mathcal{D}_2 = \operatorname{span}_{C^\infty(\mathbb{R}^3,\mathbb{R})}\left\{\begin{pmatrix}\cos x_3\\ \sin x_3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$,

c). $\mathcal{D}_3 = \operatorname{span}_{C^\infty(\mathbb{R}^5,\mathbb{R})}\left\{\begin{pmatrix}\cos x_3\\ \sin x_3\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}\sin x_3\\ \cos x_3\\0\\0\\0\end{pmatrix}, \begin{pmatrix}\cos x_3\\ \sin x_3\\0\\0\\0\end{pmatrix}\right\}$.

5. Given the distributions find a function $\lambda(x)$ such that $d\lambda \mathcal{D}(x) = 0$:

a). $\mathcal{D}_1 = \operatorname{span}_{C^\infty(\mathbb{R}^2,\mathbb{R})}\left\{\begin{pmatrix}e^{x_2}\\1\end{pmatrix}\right\}$,

b). $\mathcal{D}_2 = \operatorname{span}_{C^\infty(\mathbb{R}^3,\mathbb{R})}\left\{\begin{pmatrix}2x_3\\-1\\0\end{pmatrix}, \begin{pmatrix}-x_1\\-2x_2\\x_3\end{pmatrix}\right\}$.

Embedded, class

List 8

1. For a linear control system $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ show that the accessibility property implies controllability.

2. For a control affine system $\dot{x}_1 = u$, $\dot{x}_2 = x_1^2$ compute the distribution Δ and check the accessibility property, controllability and local controllability from $(0,0)$.

3. Check the accessibility property of controlled Euler equations

$$\begin{cases} \dot{x}_1 = a x_2 x_3 + bu \\ \dot{x}_2 = -a x_1 x_3 + cu \\ \dot{x}_3 = du. \end{cases}$$

4. Using the result of problem 3 prove controllability of controlled Euler equations.

5. Show controllability of the Brockett integrator

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 u_2 - x_2 u_1. \end{cases}$$

6. Show local controllability of the control affine system

$$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = u. \end{cases}$$

from the equilibrium point $(0,0)$.

Embedded

List 9

1. Check S-linearizability of control systems

a). $\dot{x}_1 = e^{-x_1}x_2$, $\dot{x}_2 = e^{x_1} + u$

b). $\dot{x}_1 = x_2 \cos^2 x_1$, $\dot{x}_2 = \operatorname{tg} x_1 + u$

2. check F-linearizability of control systems

a). $\dot{x}_1 = \frac{x_2}{\cos x_1}$, $\dot{x}_2 = \sin x_1 + u$

b). $\dot{x}_1 = x_2 + x_2^2 u$, $\dot{x}_2 = u$

c). $\dot{x}_1 = x_2 + e^{x_2}x_3 - e^{x_2}x_2^3$, $\dot{x}_2 = x_3 - x_2^3$, $\dot{x}_3 = 3x_2^2 x_3 - 3x_2^5 + u$

d). $\dot{x}_1 = x_2 + e^{-x_3}x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = u$

3. Show that a control system

$\dot{x}_1 = x_2$, $\dot{x}_2 = x_1 x_4^2 + \sin x_3$, $\dot{x}_3 = x_4$, $\dot{x}_4 = u$

representing the controlled ball and beam system is not feedback linearizable.

4. For a control system

$\dot{x}_1 = \sin x_2$, $\dot{x}_2 = \sin x_3$, $\dot{x}_3 = x_4^3 + u_1$, $\dot{x}_4 = x_5 + x_4^3 - x_1^{10}$, $\dot{x}_5 = u_2$

derive and solve the equivalence equations for a corresponding Brunovsky canonical form.

Embedded - cases

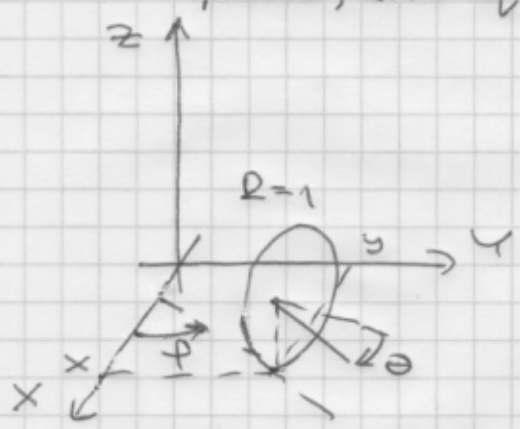
List no 10

1. For a non-redundant rigid manipulator described by the equations

$$\begin{cases} Q(q)\ddot{q} + B(q,\dot{q}) = u \\ y = k(q), \end{cases}$$

$q, u, y \in \mathbb{R}^n$; design a trajectory tracking algorithm ($y(t) \to y_d(t)$) based on the input-output decoupling and linearization.

2. Given a wheel rolling vertically on a horizontal plane, with equations



$$\dot{x} = \eta_1 \cos\varphi, \quad \dot{y} = \eta_1 \sin\varphi$$
$$\dot{\varphi} = \eta_2, \quad \dot{\theta} = \eta_1$$
$$\dot{\eta}_1 = u_2, \quad \dot{\eta}_2 = u_1$$
$$y_1 = x, \quad y_2 = \varphi$$

design a trajectory $y_d(t)$ tracking algorithm based on the input-output decoupling and linearization. Introduce new coordinates and examine the zero dynamics.

3. For a control system of the form

$$\dot{x}_1 = x_2, \ \dot{x}_2 = \sin x_3, \ \dot{x}_3 = x_4, \ \dot{x}_4 = u, \ y = x_1$$

check linearizability using the Jakubczyk-Respondek theorem and then accomplish the input-output linearization.